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MATHEMATICS AND STATISTICS

EVALUATION PLAN

1. (a) Theory / Written examination (3 hours) : **80 marks**
   (b) Practical examination (1 hour) : **20 marks**
   Total : **100 marks**

2. Question paper pattern for the theory / written examination :

Section	Question Type	Question No.	Internal Choice	Total Marks	Marks with Option
A	Multiple Choice Questions (MCQ)	Q. 1. [(i) to (viii)]	–	16	16
	Very Short Answer Questions (VSA)	Q. 2. [(i) to (iv)]	–	4	4
B	Short Answer Questions (SA) – I	Q. 3. to Q. 14.	8 out of 12 Qs.	16	24
C	Short Answer Questions (SA) – II	Q. 15. to Q. 26.	8 out of 12 Qs.	24	36
D	Long Answer Questions (LA)	Q. 27. to Q. 34.	5 out of 8 Qs.	20	32
				80	**112**

3. Chapterwise distribution of marks in the question paper :

Chapter No.	Name of the Chapter	Marks with Option
1	Mathematical Logic	8
2	Matrices	6
3	Trigonometric Functions	10
4	Pair of Straight Lines	6
5	Vectors	12
6	Line and Plane	10
7	Linear Programming (LPP)	4
8	Differentiation	9
9	Applications of Derivatives	9
10	Indefinite Integration	10
11	Definite Integration	6
12	Application of Definite Integration	4
13	Differential Equations	8
14	Probability Distribution	5
15	Binomial Distribution	5
	Total	**112**

NON-EVALUATIVE PORTION FOR THE ACADEMIC YEAR 2020-21 AS DECLARED ON 22-07-2020

MATHEMATICS Part-1

Chapter No. & Name	Page No.	Column	Matter of Self-Study
1. Mathematical Logic	14	1.3.2	Dual
	21	1.5	Applications of Logic to switching Circuit
2. Matrices	40	—	Uniqueness of Inverse of a matrix
	57	2.3.2	Method of Reduction
3. Trigonometric Functions	68	—	The General Solution : Theorem 3.1, 3.2, 3.3, 3.4, 3.5, 3.6
	82	3.3.7	Applications of Sine rule, Cosine rule, Projection rule
	94	3.4.8	Proofs of Properties of Inverse Trigonometric Functions
4. Pair of Straight Lines	124	4.4	General Second Degree equation in x and y
5. Vectors	180	5.5.1	Proof of Theorem 9 (Volume of Parallelopiped)
	180	5.5.1	Proof of Theorem 10 (Volume of tetrahedron)
	181	5.5.2	Vector Triple Product
6. Line and Plane	217	6.5.1	Angle between planes
	217	6.5.2	Angle between a line and a plane
	218	6.6	Coplanarity of two lines
	219	6.7	Distance of a point from a plane
7. Linear Programming	235	7.2.2	Mathematical formulation of L.P.P. and word problems

MATHEMATICS Part-2

Chapter No. & Name	Page No.	Column	Matter of Self-Study
1. Differentiation	56	1.5.2	Successive differentiation of some standard functions
2. Application of Derivatives	73	2.2.1	Approximations
	76	2.3.1	Rolle's Theorem
	78	2.3.2	Lagrange's Mean Value Theorem
3. Indefinite Integration	135	3.3.2	Integral of Type $\int (px+q)\sqrt{ax^2+bx+c}$
	145	3.5	Something Interesting
4. Definite Integration	151	4.1	Definite Integral as a limit of sum
	173	—	Reduction Formulae
6. Differential Equations	204	6.4.2	Linear Differential Equation
7. Probability Distribution	233	7.4	Probability Distributions of Continuous Random Variables

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PART 1 — MODEL QUESTION PAPER

(WITH SOLUTION AND MARKING SCHEME)

MATHEMATICS & STATISTICS

(ARTS & SCIENCE)

Time : 3 Hours] **[Max. Marks : 80**

General Instructions :

1. *Question paper consists of **34** questions divided into **FOUR** sections, namely **A, B, C** and **D**.*
   - *(1) **Section – A :** Q. No. 1 contains **8 multiple choice** type questions carrying **two marks** each.*
     *Q. No. 2 contains **4 very short answers** type questions carrying **one mark** each.*
   - *(2) **Section – B :** Q. No. 3 to Q. No. 14 are **12 short answer** type questions carrying **two marks** each out of which **any eight** are to be attempted.*
   - *(3) **Section – C :** Q. No. 15 to Q. No. 26 are **12 short answer** type questions carrying **three marks** each out of which **any eight** are to be attempted.*
   - *(4) **Section – D :** Q. No. 27 to Q. No. 34 are **8 long answer** type questions carrying **four marks** each out of which **any five** are to be attempted.*
2. *Figures to the right indicate full marks.*
3. *Start each section on new page.*
4. *For each MCQ, the correct answer must be written along with its alphabet :*
   e.g. (a) ....../ (b) ....../ (c) ....../ (d) ......, etc.
5. *Evaluation of each MCQ would be done for the first attempt only.*
6. *Use of graph paper is not necessary. Only rough sketch of graph is expected.*
7. *Use log table if necessary. Use of calculator is not allowed.*

SECTION – A

Q. 1. Select and write the most appropriate answer from the given alternatives for each questions : **[16]**

(i) The principal solutions of equation $\sin\theta = -\frac{1}{2}$ are

(a) $\frac{5\pi}{6}, \frac{\pi}{6}$ (b) $\frac{7\pi}{6}, \frac{11\pi}{6}$

(c) $\frac{\pi}{6}, \frac{7\pi}{6}$ (d) $\frac{7\pi}{6}, \frac{\pi}{3}$ (2)

(ii) If the volume of a parallelopiped whose coterminus edges are $-3\hat{i}+2\hat{j}+n\hat{k}$, $2\hat{i}+\hat{j}-\hat{k}$, $-\hat{i}+3\hat{j}+2\hat{k}$ is 7 cu units, then the value of n is

(a) 0 (b) 1 (c) 3 (d) 4 (2)

(iii) If the line $\frac{x}{3}=\frac{y}{4}=z$ is perpendicular to the line $\frac{x-1}{k}=\frac{y+2}{3}=\frac{z-3}{k-1}$, then the value of k is

(a) $\frac{11}{4}$ (b) $-\frac{11}{4}$ (c) $\frac{11}{2}$ (d) $\frac{4}{11}$ (2)

(iv) The equation of the plane passing through the points $(1, -1, 1)$, $(3, 2, 4)$ and parallel to Y-axis is

(a) $3x+2z-1=0$ (b) $3x-2z=1$

(c) $3x+2z+1=0$ (d) $3x+2z=2$ (2)

(v) If $y=\log_{10}\sin x$, $\frac{dy}{dx}$ is

(a) $\cot x$ (b) $\cot x \cdot \log_{10} e$ (c) $\tan x$ (d) $\log_{10}\cos x$ (2)

(vi) Let $f(x)=x^3-6x^2+9x+18$, then $f(x)$ is strictly decreasing in

(a) $(-\infty, 1)$ (b) $[3, \infty)$ (c) $(-\infty, 1]\cup[3, \infty)$ (d) $(1, 3)$ (2)

(vii) The order and degree of the differential equation $\sqrt{1+\frac{1}{\left(\frac{dy}{dx}\right)^2}}=\left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}}$ are respectively

(a) 2, 3 (b) 2, 1 (c) 1, 2 (d) 2, 2 (2)

(viii) If the d.r.v. X has the following probability distribution :

X	-2	-1	0	1	2	3
$P(X=x)$	0.1	k	0.2	$2k$	0.3	k

then $P(X=-1)=$ ....

(a) $\frac{3}{10}$ (b) $\frac{2}{10}$ (c) $\frac{1}{10}$ (d) $\frac{4}{10}$ (2)

Q. 2. Answer the following questions : **[4]**

(i) Write the following statement in symbolic form :

"Angle is neither acute nor obtuse." (1)

(ii) What is the principal value branch of $\sec^{-1}x$. (1)

(iii) Evaluate : $\int \frac{x+\sqrt{x}}{\sqrt{x+1}}\,dx$. (1)

(iv) Define the degree of the differential equation. (1)

SECTION – B

Attempt any EIGHT of the following questions : **[16]**

Q. 3. Find the truth values of the following statements :

(i) Neither 21 is a prime number nor it is divisible by 3.

(ii) $3+5>7$ if and only if $4+6<10$. (2)

Q. 4. Construct the truth table of the statement pattern $[(p \rightarrow q) \wedge q] \rightarrow p$. (2)

Q. 5. Find the adjoint of the matrix $\begin{bmatrix} 2 & -3 \\ 3 & 5 \end{bmatrix}$. (2)

Q. 6. Find k, if the sum of slopes of the lines represented by $x^2+kxy-3y^2=0$ is twice their product. (2)

Q. 7. Find the direction ratios of a line perpendicular to the two lines whose direction ratios are $-2, 1, -1$ and $-3, -4, 1$. (2)

Q. 8. If $|\bar{u}|=3$ and vector $\bar{u}$ is equally inclined to the unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$, find $\bar{u}$. (2)

Q. 9. If $\sec\left(\frac{x+y}{x-y}\right)=a^2$, show that $\frac{dy}{dx}=\frac{y}{x}$. (2)

Q. 10. Find $\frac{dy}{dx}$, if $y=\sin^{-1}\left(\sqrt{\frac{1-\cos x}{2}}\right)$. (2)

Q. 11. Prove that : $\int \operatorname{cosec} x\,dx=\log\left|\tan\left(\frac{x}{2}\right)\right|+c$. (2)

Q. 12. Evaluate : $\int_0^1 \frac{x^2-2}{x^2+1}\,dx$. (2)

Q. 13. Solve the following differential equation :

$\sec^2 x \cdot \tan y\, dx + \sec^2 y \cdot \tan x\, dy = 0.$ (2)

Q. 14. Let $X \sim B(n, p)$. If $E(X) = 5$ and $\text{Var}(X) = 2.5$, find n and p. (2)

SECTION – C

Attempt any EIGHT of the following questions : **[24]**

Q. 15. Using truth table, examine whether $[p \to (q \to r)] \leftrightarrow [(p \wedge q) \to r]$ is a tautology or a contradiction or a contingency. (3)

Q. 16. Prove that :

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - 3\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = -\frac{3\pi}{4}.$$ (3)

Q. 17. If $\bar{a}$ and $\bar{b}$ are any two non-zero, non-collinear vector lying in the same plane, then prove that any vector $\bar{r}$ is coplanar with them if and only if there exist unique scalars t_1, t_2 such that $\bar{r} = t_1\bar{a} + t_2\bar{b}$. (3)

Q. 18. Find the volume of a tetrahedron whose vertices are A$(-1, 2, 3)$, B$(3, -2, 1)$, C$(2, 1, 3)$ and D$(-1, -2, 4)$. (3)

Q. 19. Find the shortest distance between the lines

$\bar{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})$ and $\bar{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(\hat{i} + 4\hat{j} - 5\hat{k})$. (3)

Q. 20. Find the cartesian equation of the plane

$\bar{r} = (5\hat{i} - 2\hat{j} - 3\hat{k}) + \lambda(\hat{i} + \hat{j} + \hat{k}) + \mu(\hat{i} - 2\hat{j} + 3\hat{k}).$ (3)

Q. 21. If $x = f(t)$ and $y = g(t)$ are differentiable functions of t, then prove that y is a differentiable function of x and $\dfrac{dy}{dx} = \dfrac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)}$, where $\dfrac{dx}{dt} \neq 0$. (3)

Q. 22. Divide the number 20 into two parts such that sum of their squares is minimum. (3)

Q. 23. Evaluate : $\displaystyle\int \frac{3x - 2}{(x+1)^2(x+3)}\, dx.$ (3)

Q. 24. Solve the differential equation :

$y(1 + \log x)\dfrac{dx}{dy} - x\log x = 0$, when $y = e^2$, $x = e$. (3)

Q. 25. Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers. Find $E(X)$. (3)

Q. 26. A die is thrown, 6 times. If getting an odd number is a success, find the probability of (i) 5 successes (ii) at least 5 successes. (3)

SECTION – D

Attempt any FIVE of the following questions : [20]

Q. 27. If $f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that $[f(x)]^{-1} = f(-x)$. (4)

Q. 28. In $\triangle$ABC, if cot A, cot B, cot C are in A.P., then show that a^2, b^2, c^2 are also in A.P. (4)

Q. 29. If θ is the acute angle between the lines given by $ax^2 + 2hxy + by^2 = 0$, then prove that $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$, if $a + b \neq 0$.

Hence, find the acute angle between the lines $x^2 - 4xy + y^2 = 0$. (4)

Q. 30. Minimize $z = 8x + 10y$, subject to $2x + y \geqslant 7$, $2x + 3y \geqslant 15$, $y \geqslant 2$, $x \geqslant 0$, $y \geqslant 0$. (4)

Q. 31. Find the points on the curve $y = \sqrt{x-3}$ where the tangent is perpendicular to the line $6x + 3y - 5 = 0$. (4)

Q. 32. If $x = \phi(t)$ is a differentiable function of t, then prove that

$\int f(x)\,dx = \int f[\phi(t)] \cdot \phi'(t)\,dt$.

Hence, evaluate $\int \frac{x^{n-1}}{\sqrt{1+4x^n}}\,dx$. (4)

Q. 33. Evaluate : $\int_{-a}^{a} \sqrt{\frac{a-x}{a+x}}\,dx$. (4)

Q. 34. Find the area of the region in first quadrant bounded by the circle $x^2 + y^2 = 4$, the X-axis and the line $x = y\sqrt{3}$. (4)

SECTION- A

Note : Calculation, if required, may be done at the bottom of the page.

Q. 1. (i) (b) $\frac{7\pi}{6}, \frac{11\pi}{6}$ (2 marks)

(ii) (d) 4 (2 marks)

(iii) (b) $-\frac{11}{4}$ (2 marks)

(iv) (b) $3x - 2z = 1$ (2 marks)

(v) (b) $\cot x \cdot \log_{10} e$ (2 marks)

(vi) (d) (1, 3) (2 marks)

(vii) (a) 2, 3 (2 marks)

(viii) (c) $\frac{1}{10}$ (2 marks)

Rough calculation :

(i) $\sin\theta = -\frac{1}{2} = -\sin\frac{\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right) = \sin\left(2\pi - \frac{\pi}{6}\right)$

$\therefore \sin\theta = \sin\frac{7\pi}{6} = \sin\frac{11\pi}{6}$

$\therefore \theta = \frac{7\pi}{6}, \frac{11\pi}{6}.$

(ii) $\begin{vmatrix} -3 & 2 & n \\ 2 & 1 & -1 \\ -1 & 3 & 2 \end{vmatrix} = 7$

$\therefore -3(2+3) - 2(4-1) + n(6+1) = 7$

$\therefore -15 - 6 + 7n = 7$

$\therefore 7n = 28 \qquad \therefore n = 4.$

(iii) $3k + 4(3) + 1(k - 1) = 0$

$\therefore\ 3k + 12 + k - 1 = 0$

$\therefore\ 4k = -11 \qquad \therefore\ k = -\frac{11}{4}.$

(iv) Let a, b, c be the direction ratios of normal to the plane passing through $(1, -1, 1)$.

Then equation of plane is $a(x - 1) + b(y + 1) + c(z - 1) = 0$

Since it passes through (3, 2, 4), we get

$2a + 3b + 3c = 0$

Also, plane is parallel to Y-axis, $b = 0$

$\therefore\ 2a + 3c = 0 \qquad \therefore\ c = -\frac{2a}{3}$

$\therefore$ equation of the plane is

$$a(x - 1) + 0 - \frac{2a}{3}(z - 1) = 0$$

$\therefore\ 3x - 3 - 2z + 2 = 0 \qquad \therefore\ 3x - 2z = 1.$

(v) $y = \log_{10}\sin x = \frac{\log\sin x}{\log 10} = \log\sin x \cdot \log_{10}e$

$\therefore\ \frac{dy}{dx} = \log_{10}e \times \frac{1}{\sin x} \times \cos x$

$= \cot x \cdot \log_{10}e.$

(vi) $f(x) = x^3 - 6x^2 + 9x + 18$

$\therefore\ f'(x) = 3x^2 - 12x + 9$

$f(x)$ is strictly decreasing if $f'(x) < 0$

$\therefore\ 3x^2 - 12x + 9 < 0$

$\therefore\ x^2 - 4x < -3$

$\therefore\ x^2 - 4x + 4 < 1$

$\therefore\ (x - 2)^2 < 1$

$\therefore\ -1 < x - 2 < 1$

$\therefore\ 1 < x < 3 \qquad \therefore\ x \in (1, 3)$

(vii) $\sqrt{1+\frac{1}{\left(\frac{dy}{dx}\right)^2}}=\left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}}$

$\therefore\ 1+\frac{1}{\left(\frac{dy}{dx}\right)^2}=\left(\frac{d^2y}{dx^2}\right)^3$

$\therefore\ \left(\frac{d^2y}{dx^2}\right)^3\left(\frac{dy}{dx}\right)^2=\left(\frac{dy}{dx}\right)^2+1$

$\therefore$ order $=2$, degree $=3$.

(viii) $0.1+k+0.2+2k+0.3+k=1$

$\therefore\ 4k=0.4$

$\therefore\ k=0.1$

$\therefore\ P(X=-1)=k=0.1=\frac{1}{10}$.

Q. 2. (i) Let p : Angle is acute

q : Angle is obtuse

Then the symbolic form of the given statement is $\sim p \wedge \sim q$.

(1 mark)

(ii) The principal value branch of $\sec^{-1}x$ is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ (1 mark)

(iii) $$\int \frac{x+\sqrt{x}}{\sqrt{x}+1}dx = \int \frac{\sqrt{x}(\sqrt{x}+1)}{\sqrt{x}+1}dx$$

$$= \int \sqrt{x}\,dx = \int x^{\frac{1}{2}}dx$$

$$= \frac{x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + c = \frac{2}{3}x^{\frac{3}{2}} + c.$$

(1 mark)

(iv) The degree of a differential equation is the power of the highest order derivative occurring in it, when the D.E. is so written that the derivatives are free from negative or fractional indices. (1 mark)

SECTION - B

Q. 3. (i) Let p : 21 is a prime number.

q : It is divisible by 3.

Then the symbolic form of the given statement is $\sim p \wedge \sim q$.

The truth values of p and q are F and T respectively.

$\therefore$ the truth value of $\sim p \wedge \sim q$ is F. ... $[\sim F \wedge \sim T \equiv T \wedge F \equiv F]$

(1 mark)

(ii) Let p : $3 + 5 > 7$

q : $4 + 6 < 10$

Then the symbolic form of the given statement is $p \leftrightarrow q$.

The truth values of p and q are T and F respectively.

$\therefore$ truth value of $p \leftrightarrow q$ is F. ... $[T \leftrightarrow F \equiv F]$ (1 mark)

Q. 4.

1	2	3	4	5
p	q	$p \to q$	$(p \to q) \wedge q$	$[(p \to q) \wedge q] \to p$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

(Column no. 4 : 1 mark, column no. 5 : 1 mark)

Q. 5. Let $A = \begin{bmatrix} 2 & -3 \\ 3 & 5 \end{bmatrix}$

Here, $a_{11} = 2$, $M_{11} = 5$

$\therefore A_{11} = (-1)^{1+1}(5) = 5$

$a_{12} = -3$, $M_{12} = 3$

$\therefore A_{12} = (-1)^{1+2}(3) = -3$

$a_{21} = 3$, $M_{21} = -3$

$\therefore A_{21} = (-1)^{2+1}(-3) = 3$

$a_{22} = 5$, $M_{22} = 2$

$\therefore A_{22} = (-1)^{2+2} = 2$ (1 mark)

$\therefore$ the co-factor matrix $= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ 3 & 2 \end{bmatrix}$

$\therefore \text{adj } A = \begin{bmatrix} 5 & 3 \\ -3 & 2 \end{bmatrix}$ (1 mark)

Q. 6. Comparing the equation $x^2 + kxy - 3y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get, $a = 1$, $2h = k$, $b = -3$.

Let m_1 and m_2 be the slopes of the lines represented by $x^2 + kxy - 3y^2 = 0$.

$\therefore m_1 + m_2 = \dfrac{-2h}{b} = -\dfrac{k}{(-3)} = \dfrac{k}{3}$

and $m_1 m_2 = \dfrac{a}{b} = \dfrac{1}{(-3)} = -\dfrac{1}{3}$. (1 mark)

Now, $m_1 + m_2 = 2(m_1 m_2)$ ... (Given)

$\therefore \dfrac{k}{3} = 2\left(-\dfrac{1}{3}\right)$

$\therefore k = -2$. (1 mark)

Q. 7. Let a, b, c, be the direction ratios of the required line which is perpendicular to the two lines whose direction ratios are

$-2, 1, -1$ and $-3, -4, 1.$

$\therefore \quad -2a+b-c=0$

and $-3a-4b+c=0$ (1 mark)

$$\therefore \frac{a}{\begin{vmatrix} 1 & -1 \\ -4 & 1 \end{vmatrix}} = \frac{b}{-\begin{vmatrix} -2 & -1 \\ -3 & 1 \end{vmatrix}} = \frac{c}{\begin{vmatrix} -2 & 1 \\ -3 & -4 \end{vmatrix}}$$

$$\therefore \frac{a}{(1-4)} = \frac{b}{-(-2-3)} = \frac{c}{8+3}$$

$$\therefore \frac{a}{-3} = \frac{b}{5} = \frac{c}{11}$$

Hence the direction ratios of the required line are $-3, 5, 11.$

(1 mark)

Q. 8. The line of the vector $\bar{u}$ is equally inclined to the coordinate axes.

If α, β, γ are the direction angles of this line, then $\alpha = \beta = \gamma$.

$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ gives

$3\cos^2\alpha = 1 \quad \therefore \cos^2\alpha = 1/3$

$\therefore \quad \cos\alpha = \pm(1/\sqrt{3}).$ (1 mark)

$\therefore$ the unit vectors along this line are

$(1/\sqrt{3})\hat{i} + (1/\sqrt{3})\hat{j} + (1/\sqrt{3})\hat{k}$ and

$(-1/\sqrt{3})\hat{i} + (-1/\sqrt{3})\hat{j} + (-1/\sqrt{3})\hat{k}$

i.e. $(1/\sqrt{3})(\hat{i}+\hat{j}+\hat{k})$ and $(-1/\sqrt{3})(\hat{i}+\hat{j}+\hat{k})$

Now, $|\bar{u}| = 3$

$\therefore \quad \bar{u} = \pm 3(1/\sqrt{3})(\hat{i}+\hat{j}+\hat{k})$

i.e. $\bar{u} = \pm\sqrt{3}\,(\hat{i}+\hat{j}+\hat{k}).$ (1 mark)

Q. 9. $\sec\left(\frac{x+y}{x-y}\right)=a^2$

$\therefore \frac{x+y}{x-y}=\sec^{-1}a^2$

$\therefore \frac{d}{dx}\left(\frac{x+y}{x-y}\right)=\frac{d}{dx}(\sec^{-1}a^2)$

$\therefore \frac{(x-y)\cdot\frac{d}{dx}(x+y)-(x+y)\cdot\frac{d}{dx}(x-y)}{(x-y)^2}=0$ (1 mark)

$\therefore (x-y)\left(1+\frac{dy}{dx}\right)-(x+y)\left(1-\frac{dy}{dx}\right)=0$

$\therefore (x-y)+(x-y)\left(\frac{dy}{dx}\right)-(x+y)+(x+y)\left(\frac{dy}{dx}\right)=0$

$\therefore (x-y+x+y)\frac{dy}{dx}=-x+y+x+y$

$\therefore 2x\frac{dy}{dx}=2y$

$\therefore x\frac{dy}{dx}=y$

$\therefore \frac{dy}{dx}=\frac{y}{x}.$ (1 mark)

Q. 10. $y=\sin^{-1}\left(\sqrt{\frac{1-\cos x}{2}}\right)$

$=\sin^{-1}\sqrt{\frac{2\sin^2\left(\frac{x}{2}\right)}{2}}$

$=\sin^{-1}\left[\sin\left(\frac{x}{2}\right)\right]=\frac{x}{2}$ (1 mark)

$\therefore \frac{dy}{dx}=\frac{d}{dx}\left(\frac{x}{2}\right)=\frac{1}{2}\frac{d}{dx}(x)$

$=\frac{1}{2}\times 1=\frac{1}{2}.$ (1 mark)

Q. 11. Let $I = \int \operatorname{cosec} x \, dx = \int \frac{\operatorname{cosec} x (\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x} dx$

Put $\operatorname{cosec} x - \cot x = t$

$\therefore \; (-\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x) \, dx = dt$

$\therefore \; \operatorname{cosec} x (\operatorname{cosec} x - \cot x) \, dx = dt$

$\therefore \; I = \int \frac{1}{t} dt = \log |t| + c$

$= \log |\operatorname{cosec} x - \cot x| + c$ (1 mark)

$= \log \left| \frac{1}{\sin x} - \frac{\cos x}{\sin x} \right| + c$

$= \log \left| \frac{1 - \cos x}{\sin x} \right| + c$

$= \log \left| \frac{2 \sin^2 \left(\frac{x}{2} \right)}{2 \sin \left(\frac{x}{2} \right) \cos \left(\frac{x}{2} \right)} \right| + c$

$= \log \left| \tan \left(\frac{x}{2} \right) \right| + c$

$\therefore \; \int \operatorname{cosec} x \, dx = \log \left| \tan \left(\frac{x}{2} \right) \right| + c$ (1 mark)

Q. 12. $\int_0^1 \frac{x^2 - 2}{x^2 + 1} dx = \int_0^1 \frac{(x^2 + 1) - 3}{x^2 + 1} dx$

$= \int_0^1 \left(1 - \frac{3}{x^2 + 1} \right) dx$

$= [x - 3 \tan^{-1} x]_0^1$ (1 mark)

$= (1 - 3 \tan^{-1} 1) - (0 - 3 \tan^{-1} 0)$

$= 1 - 3 \left(\frac{\pi}{4} \right) - 0 = 1 - \frac{3\pi}{4}.$ (1 mark)

Q. 13. $\sec^2 x \cdot \tan y \, dx + \sec^2 y \cdot \tan x \, dy = 0$

$\therefore \dfrac{\sec^2 x}{\tan x} dx + \dfrac{\sec^2 y}{\tan y} dy = 0$

On integrating, we get

$\int \dfrac{\sec^2 x}{\tan x} dx + \int \dfrac{\sec^2 y}{\tan y} dy = c_1$ (1 mark)

Each of these integrals is of the type $\int \dfrac{f'(x)}{f(x)} dx = \log |f(x)| + c$

$\therefore$ the general solution is $\log|\tan x| + \log|\tan y| = \log c$,

where $c_1 = \log c$

$\therefore \log|\tan x \cdot \tan y| = \log c$

$\therefore \tan x \cdot \tan y = c.$ (1 mark)

Q. 14. Given : $E(X) = 5$ and $Var(X) = 2.5$

$\therefore np = 5$ and $npq = 2.5$

$\therefore \dfrac{npq}{np} = \dfrac{2.5}{5}$

$\therefore q = 0.5 = \dfrac{5}{10} = \dfrac{1}{5}$

$\therefore p = 1 - q = 1 - \dfrac{1}{2} = \dfrac{1}{2}$ (1 mark)

Substituting $p = \dfrac{1}{2}$ in $np = 5$, we get

$n\left(\dfrac{1}{2}\right) = 5 \qquad \therefore n = 10$

Hence, $n = 10$ and $p = \dfrac{1}{2}$. (1 mark)

SECTION - C

Q. 15.

1	2	3	4	5	6	7	8
p	q	r	$q \to r$	$p \to (q \to r)$	$p \wedge q$	$(p \wedge q) \to r$	$[p \to (q \to r)] \leftrightarrow [(p \wedge q) \to r]$
T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	T
T	F	T	T	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	F	T	T
F	T	F	F	T	F	T	T
F	F	T	T	T	F	T	T
F	F	F	T	T	F	T	T

(Column no. 5 : 1 mark; Column no. 8 : 1 mark)

All the entries in the last column of the above truth table are T.

$\therefore$ $[p \to (q \to r)] \leftrightarrow [(p \wedge q) \to r]$ is a tautology. (Conclusion : 1 mark)

Q. 16.

Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \alpha$, where $-\frac{\pi}{2} \leqslant \alpha \leqslant \frac{\pi}{2}$

$\therefore \sin \alpha = \frac{1}{\sqrt{2}} = \sin \frac{\pi}{4}$

$\therefore \alpha = \frac{\pi}{4}$ ... $\left[\because -\frac{\pi}{2} \leqslant \frac{\pi}{4} \leqslant \frac{\pi}{2}\right]$

$\therefore \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ ... (1) (1 mark)

Let $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \beta$, where $-\frac{\pi}{2} \leqslant \beta \leqslant \frac{\pi}{2}$

$\therefore \sin \beta = \frac{\sqrt{3}}{2} = \sin \frac{\pi}{3}$

$\therefore \beta = \frac{\pi}{3}$ ... $\left[\because -\frac{\pi}{2} \leqslant \frac{\pi}{3} \leqslant \frac{\pi}{2}\right]$

$\therefore \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ ... (2) (1 mark)

$$LHS = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - 3\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\pi}{4} - 3\left(\frac{\pi}{3}\right) \quad \text{... [By (1) and (2)]}$$

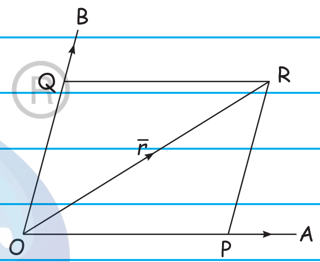
$$= \frac{\pi}{4} - \pi = -\frac{3\pi}{4} = RHS.$$ (1 mark)

Q. 17. Let $\bar{a}, \bar{b}, \bar{r}$ be coplanar.

Take any point O in the plane of $\bar{a}, \bar{b}$ and $\bar{r}$.

Represents the vectors $\bar{a}, \bar{b}$ and $\bar{r}$ by $\overline{OA}, \overline{OB}$ and $\overline{OR}$.

Take the point P on $\bar{a}$ and Q on $\bar{b}$ such that OPRQ is a parallelogram.



Now $\overline{OP}$ and $\overline{OA}$ are collinear vectors.

$\therefore$ there exists a non-zero scalar t_1 such that

$$\overline{OP} = t_1 \cdot \overline{OA} = t_1 \cdot \bar{a}.$$

Also $\overline{OQ}$ and $\overline{OB}$ are collinear vectors.

$\therefore$ there exists a non-zero scalar t_2 such that

$$\overline{OQ} = t_2 \cdot \overline{OB} = t_2 \cdot \bar{b}.$$

Now, by parallelogram law of addition of vectors,

$\overline{OR} = \overline{OP} + \overline{OQ} \quad \therefore \bar{r} = t_1\bar{a} + t_2\bar{b}$

Thus, $\bar{r}$ is expressed as a linear combination $t_1\bar{a} + t_2\bar{b}$ (1 mark)

<u>Uniqueness :</u>

Let, if possible, $\bar{r} = t_1'\bar{a} + t_2'\bar{b}$, where t_1', t_2' are non-zero scalars.

Then, $t_1\bar{a} + t_2\bar{b} = t_1'\bar{a} + t_2'\bar{b}$

$\therefore (t_1 - t_1')\bar{a} = -(t_2 - t_2')\bar{b}$ ... (1)

We want to show that $t_1 = t_1'$ and $t_2 = t_2'$.

Suppose $t_1 \neq t_1'$, i.e. $t_1 - t_1' \neq 0$ and $t_2 \neq t_2'$, i.e. $t_2 - t_2' \neq 0$.

Then dividing both sides of (1) by $t_1 - t_1'$, we get

$$\bar{a} = -\left(\frac{t_2 - t_2'}{t_1 - t_1'}\right)\bar{b}$$

This shows that the vector $\overline{a}$ is a non-zero scalar multiple of $\overline{b}$.

$\therefore$ $\overline{a}$ and $\overline{b}$ are collinear vectors.

This is a contradiction, since $\overline{a}$, $\overline{b}$ are given to be non-collinear.

$\therefore$ $t_1 = t_1'$

Similarly, we can show that $t_2 = t_2'$.

This shows that $\overline{r}$ is uniquely expressed as a linear combination $t_1\overline{a} + t_2\overline{b}$. (1 mark)

Conversely :

Let $\overline{r} = t_1\overline{a} + t_2\overline{b}$, where t_1, t_2 are scalars.

Since, $\overline{a}$, $\overline{b}$ are coplanar, $t_1\overline{a}$, $t_2\overline{b}$ are also coplanar

$\therefore$ $\overline{r} = t_1\overline{a} + t_2\overline{b}$ is coplanar with $\overline{a}$ and $\overline{b}$. (1 mark)

Q. 18. The position vectors $\overline{a}$, $\overline{b}$, $\overline{c}$ and $\overline{d}$ of the points A, B, C and D w.r.t. the origin are

$\overline{a} = -\hat{i} + 2\hat{j} + 3\hat{k}$, $\overline{b} = 3\hat{i} - 2\hat{j} + \hat{k}$, $\overline{c} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\overline{d} = -\hat{i} - 2\hat{j} + 4\hat{k}$.

$$\therefore \overline{AB} = \overline{b} - \overline{a} = (3\hat{i} - 2\hat{j} + \hat{k}) - (-\hat{i} + 2\hat{j} + 3\hat{k})$$
$$= 4\hat{i} - 4\hat{j} - 2\hat{k}$$
$$\overline{AC} = \overline{c} - \overline{a} = (2\hat{i} + \hat{j} + 3\hat{k}) - (-\hat{i} + 2\hat{j} + 3\hat{k})$$
$$= 3\hat{i} - \hat{j}$$

and
$$\overline{AD} = \overline{d} - \overline{a} = (-\hat{i} - 2\hat{j} + 4\hat{k}) - (-\hat{i} + 2\hat{j} + 3\hat{k})$$
$$= -4\hat{j} + \hat{k}$$
(1 mark)

$$\therefore [\overline{AB}\ \overline{AC}\ \overline{AD}] = \begin{vmatrix} 4 & -4 & -2 \\ 3 & -1 & 0 \\ 0 & -4 & 1 \end{vmatrix}$$

$$= 4(-1+0) + 4(3-0) - 2(-12+0)$$
$$= -4 + 12 + 24 = 32$$
(1 mark)

$$\therefore \text{ volume of the tetrahedron} = \frac{1}{6}\left|[\overline{AB}\ \overline{AC}\ \overline{AD}]\right|$$
$$= \frac{1}{6}(32) = \frac{16}{3} \text{ cu units.}$$
(1 mark)

Q. 19. We know that the shortest distance between the skew lines $\bar{r} = \bar{a_1} + \lambda \bar{b_1}$ and $\bar{r} = \bar{a_2} + \mu \bar{b_2}$ is given by

$$d = \left| \frac{(\bar{a_2} - \bar{a_1}) \cdot (\bar{b_1} \times \bar{b_2})}{|\bar{b_1} \times \bar{b_2}|} \right|.$$ (1 mark)

Here, $\bar{a_1} = 4\hat{i} - \hat{j}$, $\bar{a_2} = \hat{i} - \hat{j} + 2\hat{k}$, $\bar{b_1} = \hat{i} + 2\hat{j} - 3\hat{k}$, $\bar{b_2} = \hat{i} + 4\hat{j} - 5\hat{k}$.

$$\therefore \bar{b_1} \times \bar{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 1 & 4 & -5 \end{vmatrix}$$

$$= (-10 + 12)\hat{i} - (-5 + 3)\hat{j} + (4 - 2)\hat{k}$$

$$= 2\hat{i} + 2\hat{j} + 2\hat{k}$$

and $\bar{a_2} - \bar{a_1} = (\hat{i} - \hat{j} + 2\hat{k}) - (4\hat{i} - \hat{j}) = -3\hat{i} + 2\hat{k}$ (1 mark)

$$\therefore (\bar{a_2} - \bar{a_1}) \cdot (\bar{b_1} \times \bar{b_2}) = (-3\hat{i} + 2\hat{k}) \cdot (2\hat{i} + 2\hat{j} + 2\hat{k})$$

$$= -3(2) + 0(2) + 2(2)$$

$$= -6 + 0 + 4 = -2$$

and $|\bar{b_1} \times \bar{b_2}| = \sqrt{2^2 + 2^2 + 2^2}$

$$= \sqrt{4 + 4 + 4} = 2\sqrt{3}$$

$\therefore$ required shortest distance between the given lines

$$= \left| \frac{-2}{2\sqrt{3}} \right| = \frac{1}{\sqrt{3}} \text{ units.}$$ (1 mark)

Q. 20. The equation $\bar{r} = \bar{a} + \lambda \bar{b} + \mu \bar{c}$ represents a plane passing through a point having position vector $\bar{a}$ and parallel to vectors $\bar{b}$ and $\bar{c}$.

Here, $\bar{a} = 5\hat{i} - 2\hat{j} - 3\hat{k}$, $\bar{b} = \hat{i} + \hat{j} + \hat{k}$, $\bar{c} = \hat{i} - 2\hat{j} + 3\hat{k}$

$$\therefore \bar{b} \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= (3 + 2)\hat{i} - (3 - 1)\hat{j} + (-2 - 1)\hat{k}$$

$$= 5\hat{i} - 2\hat{j} - 3\hat{k} = \bar{a}$$

Also, $\bar{a} \cdot (\bar{b} \times \bar{c}) = \bar{a} \cdot \bar{a} = |\bar{a}|^2$

$$= (5)^2 + (-2)^2 + (-3)^2 = 38$$ (1 mark)

The vector equation of the plane passing through $A(\bar{a})$ and parallel to $\bar{b}$ and $\bar{c}$ is

$\bar{r}\cdot(\bar{b}\times\bar{c}) = \bar{a}\cdot(\bar{b}\times\bar{c})$

$\therefore$ the vector equation of the given plane is $\bar{r}\cdot(5\hat{i}-2\hat{j}-3\hat{k}) = 38$

(1 mark)

If $\bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$, then this equation becomes

$(x\hat{i}+y\hat{j}+z\hat{k})\cdot(5\hat{i}-2\hat{j}-3\hat{k}) = 38$

$\therefore\ 5x-2y-3z = 38.$

This is the cartesian equation of the required plane. (1 mark)

Q. 21. Let δx and δy be the increments in x and y respectively, corresponding to the increment δt in t.

Since x and y are differentiable functions of t,

$$\frac{dx}{dt} = \lim_{\delta t\to 0}\frac{\delta x}{\delta t} \text{ and } \frac{dy}{dt} = \lim_{\delta t\to 0}\frac{\delta y}{\delta t} \quad \dots (1)$$

Also, as $\delta t \to 0$, $\delta x \to 0$ ... (2) (1 mark)

$$\text{Now, } \frac{\delta y}{\delta x} = \frac{(\delta y/\delta t)}{(\delta x/\delta t)} \quad \dots (\delta t \ne 0)$$

Taking limits as $\delta t \to 0$, we get,

$$\lim_{\delta t\to 0}\frac{\delta y}{\delta x} = \lim_{\delta t\to 0}\frac{(\delta y/\delta t)}{(\delta x/\delta t)}$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \frac{\lim\limits_{\delta t\to 0}(\delta y/\delta t)}{\lim\limits_{\delta t\to 0}(\delta x/\delta t)} = \frac{(dy/dt)}{(dx/dt)} \quad \dots [\text{By (1) and (2)}]$$

(1 mark)

$\because$ the limits in RHS exist

$\therefore \lim\limits_{\delta x\to 0}\frac{\delta y}{\delta x}$ exists and is equal to $\frac{dy}{dx}$

$\therefore$ y is differentiable function of x and

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}, \text{ where } \frac{dx}{dt} \ne 0.$$ (1 mark)

Q. 22. Let the first part of 20 be x.

Then the second part is $20 - x$.

$\therefore$ sum of their squares $= x^2 + (20 - x)^2 = f(x)$ ... (Say)

(1 mark)

$\therefore \ f'(x) = \frac{d}{dx}[x^2 + (20 - x)^2]$

$= 2x + 2(20 - x) \cdot \frac{d}{dx}(20 - x)$

$= 2x + 2(20 - x) \times (0 - 1)$

$= 2x - 40 + 2x = 4x - 40$

and $f''(x) = \frac{d}{dx}(4x - 40) = 4 \times 1 - 0 = 4$ (1 mark)

The root of the equation $f'(x) = 0$,

i.e. $4x - 40 = 0$ is $x = 10$ and $f''(10) = 4 > 0$

$\therefore$ by the second derivative test, f is minimum at $x = 10$.

Hence, the required parts of 20 are 10 and 10. (1 mark)

Q. 23. Let $I = \int \frac{3x - 2}{(x + 1)^2(x + 3)} dx$

Let $\frac{3x - 2}{(x + 1)^2(x + 3)} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{x + 3}$

Then $3x - 2 = A(x + 1)(x + 3) + B(x + 3) + C(x + 1)^2$

Put $x + 1 = 0$, i.e., $x = -1$, we get

$-3 - 2 = A(0)(2) + B(2) + C(0)$

$\therefore \ -5 = 2B$

$\therefore \ B = -\frac{5}{2}$

Put $x + 3 = 0$, i.e., $x = -3$, we get

$-9 - 2 = A(-2)(0) + B(0) + C(-2)^2$

$\therefore \ -11 = 4C$

$\therefore \ C = -\frac{11}{4}$

Put $x = 0$, we get

$$-2 = A(1)(3) + B(3) + C(1)$$

$$\therefore \quad -2 = 3A + 3B + C$$

$$\therefore \quad -2 = 3A - \frac{15}{2} - \frac{11}{4}$$

$$\therefore \; 3A = -2 + \frac{15}{2} + \frac{11}{4} = \frac{-8 + 30 + 11}{4} = \frac{33}{4}$$

$$\therefore \; A = \frac{11}{4}$$ (1 mark)

$$\therefore \; \frac{3x - 2}{(x+1)^2(x+3)} = \frac{\left(\frac{11}{4}\right)}{x+1} + \frac{\left(-\frac{5}{2}\right)}{(x+1)^2} + \frac{\left(-\frac{11}{4}\right)}{x+3}$$

$$\therefore \; I = \int \left[\frac{\left(\frac{11}{4}\right)}{x+1} + \frac{\left(-\frac{5}{2}\right)}{(x+1)^2} + \frac{\left(-\frac{11}{4}\right)}{x+3} \right] dx$$

$$= \frac{11}{4} \int \frac{1}{x+1} dx - \frac{5}{2} \int (x+1)^{-2} dx - \frac{11}{4} \int \frac{1}{x+3} dx$$ (1 mark)

$$= \frac{11}{4} \log |x+1| - \frac{5}{2} \cdot \frac{(x+1)^{-1}}{-1} \cdot \frac{1}{1} - \frac{11}{4} \log |x+3| + c$$

$$= \frac{11}{4} \log |x+1| + \frac{5}{2(x+1)} - \frac{11}{4} \log |x+3| + c$$

$$= \frac{11}{4} \log \left| \frac{x+1}{x+3} \right| + \frac{5}{2(x+1)} + c.$$ (1 mark)

Q. 24. $y(1+\log x)\dfrac{dx}{dy} - x\log x = 0$

$\therefore \dfrac{1+\log x}{x\log x}\,dx - \dfrac{dy}{y} = 0$

Integrating both sides, we get

$\therefore \int \dfrac{1+\log x}{x\log x}\,dx - \int \dfrac{dy}{y} = c_1$ ... (1) (1 mark)

Put $x \log x = t$.

Then $\left[x\cdot\dfrac{d}{dx}(\log x) + (\log x)\cdot\dfrac{d}{dx}(x)\right]dx = dt$

$\therefore \left[\dfrac{x}{x} + (\log x)(1)\right]dx = dt$

$\therefore (1+\log x)\,dx = dt$

$\therefore \int \dfrac{1+\log x}{x\log x}\,dx = \int \dfrac{dt}{t} = \log|t| = \log|x\log x|$

$\therefore$ from (1), the general solution is

$\log|x\log x| - \log|y| = \log c$, where $c_1 = \log c$

$\therefore \log\left|\dfrac{x\log x}{y}\right| = \log c$

$\therefore \dfrac{x\log x}{y} = c$

$\therefore x\log x = cy$ (1 mark)

This is the general solution.

Now, $y = e^2$, when $x = e$

$\therefore e\log e = c\cdot e^2 \qquad 1 = c\cdot e$ ... $[\because \log e = 1]$

$\therefore c = \dfrac{1}{e}$

$\therefore$ the particular solution is $x\log x = \left(\dfrac{1}{e}\right)y$

$\therefore y = ex\log x.$ (1 mark)

Q. 25. Two numbers are chosen from the first 6 positive integers.

$\therefore\ n(S) = {}^6C_2 = \frac{6 \times 5}{1 \times 2} = 15$

Let X denote the larger of the two numbers.

Then X can take values 2, 3, 4, 5, 6.

When $X = 2$, the other positive number which is less than 2 is 1.

$\therefore\ n(X) = 1$

$\therefore\ P(X = 2) = P(2) = \frac{n(X)}{n(S)} = \frac{1}{15}$ (1 mark)

When $X = 3$, the other positive number less than 3 can be 1 or 2 and hence can be chosen in 2 ways.

$\therefore\ n(X) = 2$

$\therefore\ P(X = 3) = P(3) = \frac{n(X)}{n(S)} = \frac{2}{15}$

Similarly, $P(X = 4) = P(4) = \frac{3}{15}$

$P(X = 5) = P(5) = \frac{4}{15}$

$P(X = 6) = P(6) = \frac{5}{15}$ (1 mark)

$\therefore\ E(X) = \Sigma\, x_i P(x_i)$

$$= 2 \times \frac{1}{15} + 3 \times \frac{2}{15} + 4 \times \frac{3}{15} + 5 \times \frac{4}{15} + 6 \times \frac{5}{15}$$

$$= \frac{2 + 6 + 12 + 20 + 30}{15}$$

$$= \frac{70}{15} = \frac{14}{3}.$$ (1 mark)

Q. 26. Let X = number of successes, i.e., number of odd numbers

p = probability of getting an odd number in a single throw of a die

$\therefore\ p = \frac{3}{6} = \frac{1}{2}$

and $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

Given : $n = 6$

$\therefore\ X \sim B\left(6, \frac{1}{2}\right)$

The p.m.f. of X is given by

$P(X = x) = {}^{n}C_{x}\, p^{x} q^{n-x}$

i.e., $p(x) = {}^{6}C_{x}\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{6-x}$

$= {}^{6}C_{x}\left(\frac{1}{2}\right)^{6},\ x = 0, 1, 2, ..., 6$ (1 mark)

(i) $P(5 \text{ successes}) = P(X = 5)$

$= p(5) = {}^{6}C_{5}\left(\frac{1}{2}\right)^{6}$

$= {}^{6}C_{1} \times \frac{1}{64}$ ... $[\because\ {}^{n}C_{x} = {}^{n}C_{n-x}]$

$= \frac{6}{64} = \frac{3}{32}$

Hence, the probability of 5 successes is $\frac{3}{32}$. (1 mark)

(ii) $P(\text{at least 5 successes}) = P[X \geqslant 5]$

$= p(5) + p(6)$

$= {}^{6}C_{5}\left(\frac{1}{2}\right)^{6} + {}^{6}C_{6}\left(\frac{1}{2}\right)^{6}$

$= ({}^{6}C_{5} + {}^{6}C_{6})\left(\frac{1}{2}\right)^{6} = (6 + 1)\frac{1}{64} = \frac{7}{64}$

Hence, the probability of at least 5 successes is $\frac{7}{64}$. (1 mark)

SECTION - D

Q. 27.

$$f(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \ |f(x)| = \begin{vmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos x(\cos x - 0) + \sin x(\sin x - 0) + 0$$

$$= \cos^2 x + \sin^2 x = 1 \neq 0$$

$\therefore \ [f(x)]^{-1}$ exists

Consider $f(x) \cdot [f(x)]^{-1} = I$

$$\therefore \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot [f(x)]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $\cos x \times R_1$, we get

$$\begin{bmatrix} \cos^2 x & -\sin x \cos x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot [f(x)]^{-1} = \begin{bmatrix} \cos x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (1 mark)

By $R_1 + \sin x \times R_2$, we get,

$$\begin{bmatrix} 1 & 0 & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot [f(x)]^{-1} = \begin{bmatrix} \cos x & \sin x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 - \sin x \times R_1$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot [f(x)]^{-1} = \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x \cos x & \cos^2 x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (1 mark)

By $\left(\frac{1}{\cos x}\right) R_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot [f(x)]^{-1} = \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \ [f(x)]^{-1} = \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{... (1)}$$ (1 mark)

$$\text{Also, } f(-x) = \begin{bmatrix} \cos(-x) & -\sin(-x) & 0 \\ \sin(-x) & \cos(-x) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{... (2)}$$

From (1) and (2), we get

$[f(x)]^{-1} = f(-x)$. (1 mark)

Q. 28. By the sine rule,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$$

$\therefore \ \sin A = ka, \ \sin B = kb, \ \sin C = kc$ ... (1)

Now, $\cot A$, $\cot B$, $\cot C$ are in A.P.

$\therefore \ \cot C - \cot B = \cot B - \cot A$

$\therefore \ \cot A + \cot C = 2\cot B$

$$\therefore \ \frac{\cos A}{\sin A} + \frac{\cos C}{\sin C} = 2\cot B$$

$$\therefore \ \frac{\sin C \cos A + \sin A \cos C}{\sin A \cdot \sin C} = 2\cot B$$

$$\therefore \ \frac{\sin(A+C)}{\sin A \cdot \sin C} = 2\cot B$$ (1 mark)

$$\therefore \ \frac{\sin(\pi - B)}{\sin A \cdot \sin C} = 2\cot B \quad \text{... } [\because \ A+B+C=\pi]$$

$$\therefore \ \frac{\sin B}{\sin A \cdot \sin C} = 2\frac{\cos B}{\sin B}$$

$$\therefore \ \frac{\sin^2 B}{\sin A \cdot \sin C} = 2\cos B$$ (1 mark)

$$\therefore \ \frac{k^2 b^2}{(ka)(kc)} = 2\left(\frac{a^2+c^2-b^2}{2ac}\right)$$

$\therefore \dfrac{b^2}{ac} = \dfrac{a^2+c^2-b^2}{ac}$ (1 mark)

$\therefore b^2 = a^2 + c^2 - b^2$

$\therefore 2b^2 = a^2 + c^2$

Hence, a^2, b^2, c^2 are in A.P. (1 mark)

Q. 29. Let m_1 and m_2 be the slopes of the lines represented by the equation

$ax^2 + 2hxy + by^2 = 0.$ ... (1)

Then their separate equations are $y = m_1 x$ and $y = m_2 x$

$\therefore$ their combined equation is $(m_1 x - y)(m_2 x - y) = 0$

i.e. $m_1 m_2 x^2 - (m_1 + m_2)\, xy + y^2 = 0$ ... (2)

Since (1) and (2) represent the same two lines, comparing the coefficients, we get,

$\dfrac{m_1 m_2}{a} = \dfrac{-(m_1+m_2)}{2h} = \dfrac{1}{b}$

$\therefore m_1 + m_2 = -\dfrac{2h}{b}$ and $m_1 m_2 = \dfrac{a}{b}$ (1 mark)

$\therefore (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1 m_2$

$= \dfrac{4h^2}{b^2} - \dfrac{4a}{b} = \dfrac{4(h^2 - ab)}{b^2}$

$\therefore |m_1 - m_2| = \left|\dfrac{2\sqrt{h^2-ab}}{b}\right|$ (1 mark)

If θ is the acute angle between the lines, then

$\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$, if $m_1 m_2 \neq -1$

$= \left|\dfrac{(2\sqrt{h^2-ab})/b}{1 + (a/b)}\right|$, if $\dfrac{a}{b} \neq -1$

$\therefore \tan\theta = \left|\dfrac{2\sqrt{h^2-ab}}{a+b}\right|$, if $a + b \neq 0.$ (1 mark)

Comparing the equation $x^2 - 4xy + y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get

$a = 1$, $2h = -4$, i.e. $h = -2$ and $b = 1$

Let θ be the acute angle between the lines.

Then $\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$

$$= \left|\frac{2\sqrt{(-2)^2 - 1(1)}}{1+1}\right| = \left|\frac{2\sqrt{3}}{2}\right|$$

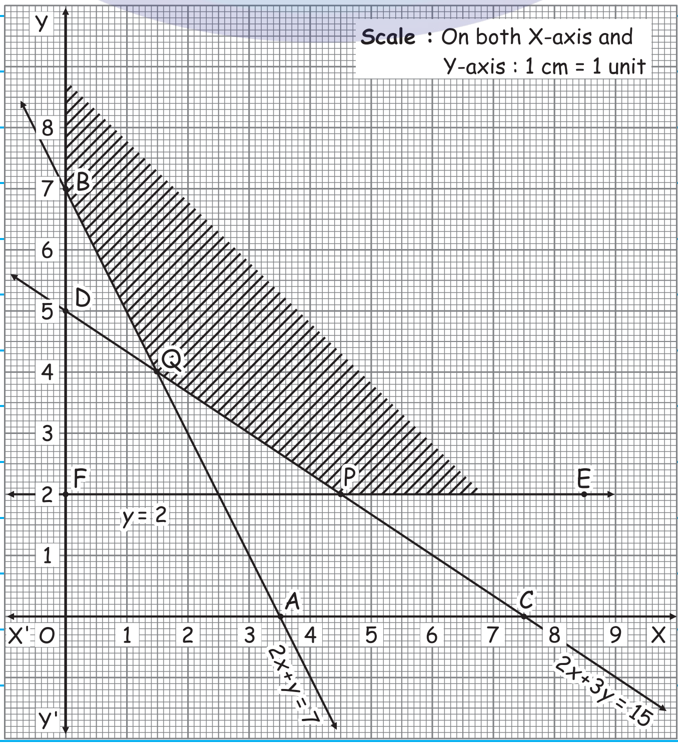
$\therefore \ \tan\theta = \sqrt{3} = \tan 60°$

$\therefore \ \theta = 60°$. (1 mark)

Q. 30. First we draw the lines AB, CD and EF whose equations are $2x + y = 7$, $2x + 3y = 15$ and $y = 2$ respectively.

Line	Equation	Points on the X-axis	Points on the Y-axis	Sign	Region
AB	$2x + y = 7$	A (3.5, 0)	B (0, 7)	$\geqslant$	non-origin side of line AB
CD	$2x + 3y = 15$	C (7.5, 0)	D (0, 5)	$\geqslant$	non-origin side of line CD
EF	$y = 2$	—	F (0, 2)	$\geqslant$	non-origin side of line EF

(1 mark)



(1 mark)

The feasible region is EPQBY which is shaded in the graph.

The vertices of the feasible region are P, Q and B (0, 7).

P is the point of intersection of the lines $2x + 3y = 15$ and $y = 2$.

Substituting $y = 2$ in $2x + 3y = 15$, we get

$2x + 3(2) = 15$

$\therefore\ 2x = 9$

$\therefore\ x = 4.5$

$\therefore\ P = (4.5, 2)$

Q is the point of intersection of the lines

$2x + 3y = 15$ ... (1)

and $2x + y = 7$ ... (2)

On subtracting, we get

$2y = 8$

$\therefore\ y = 4$

$\therefore$ from (2), $2x + 4 = 7$

$\therefore\ 2x = 3$

$\therefore\ x = 1.5$

$\therefore\ Q = (1.5, 4)$ (1 mark)

The values of the objective function $z = 8x + 10y$ at these vertices are

$z(P) = 8(4.5) + 10(2) = 36 + 20 = 56$

$z(Q) = 8(1.5) + 10(4) = 12 + 40 = 52$

$z(B) = 8(0) + 10(7) = 70$

$\therefore$ z has minimum value 52, when $x = 1.5$ and $y = 4$. (1 mark)

Q. 31. Let the required point on the curve $y = \sqrt{x-3}$ be $P(x_1, y_1)$.

Differentiating $y = \sqrt{x-3}$ w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sqrt{x-3}) = \frac{1}{2\sqrt{x-3}} \cdot \frac{d}{dx}(x-3)$$

$$= \frac{1}{2\sqrt{x-3}} \times (1-0) = \frac{1}{2\sqrt{x-3}}$$ (1 mark)

$\therefore$ slope of the tangent at (x_1, y_1)

$$= \left(\frac{dy}{dx}\right)_{\text{at }(x_1, y_1)} = \frac{1}{2\sqrt{x_1 - 3}}$$

Since this tangent is perpendicular to $6x + 3y - 5 = 0$

whose slope is $-\frac{6}{3} = -2$,

Slope of the tangent $= \frac{-1}{-2} = \frac{1}{2}$ (1 mark)

$\therefore \frac{1}{2\sqrt{x_1 - 3}} = \frac{1}{2}$

$\therefore \sqrt{x_1 - 3} = 1$

$\therefore x_1 - 3 = 1 \qquad \therefore x_1 = 4$ (1 mark)

Since (x_1, y_1) lies on $y = \sqrt{x - 3}$, we get $y_1 = \sqrt{x_1 - 3}$

When $x_1 = 4$, $y_1 = \sqrt{4 - 3} = \pm 1$

Hence, the required points are (4, 1) and (4, -1). (1 mark)

Q. 32. $x = \phi(t)$ is differentiable function of t.

$\therefore \frac{dx}{dt} = \phi'(t)$

Let $\int f(x)\,dx = F(x)$.

$\therefore \frac{d}{dx}[F(x)] = f(x)$ (1 mark)

$\therefore$ by the chain rule,

$$\frac{d}{dt}[F(x)] = \frac{d}{dx}[F(x)] \cdot \frac{dx}{dt}$$

$$= f(x) \cdot \frac{dx}{dt}$$

$$= f[\phi(t)] \cdot \phi'(t)$$ (1 mark)

$\therefore$ by the definition of integral,

$F(x) = \int f[\phi(t)] \cdot \phi'(t)\,dt$

$\therefore \int f(x)\,dx = \int f[\phi(t)] \cdot \phi'(t)\,dt$. (1 mark)

$$\text{Let } I = \int \frac{x^{n-1}}{\sqrt{1+4x^n}}\,dx$$

Put $x^n = t$

$\therefore\ nx^{n-1}dx = dt$

$$\therefore\ x^{n-1}dx = \frac{dt}{n}$$

$$\therefore\ I = \int \frac{1}{\sqrt{1+4t}}\cdot\frac{dt}{n} = \frac{1}{n}\int (1+4t)^{-\frac{1}{2}}dt$$

$$= \frac{1}{n}\cdot\frac{(1+4t)^{\frac{1}{2}}}{1/2}\times\frac{1}{4}+c$$

$$= \frac{1}{2n}\cdot\sqrt{1+4x^n}+c.$$ (1 mark)

Q. 33. $$\text{Let } I = \int_{-a}^{a}\sqrt{\frac{a-x}{a+x}}\,dx$$

$$= \int_{-a}^{a}\sqrt{\frac{(a-x)(a-x)}{(a+x)(a-x)}}\,dx$$

$$= \int_{-a}^{a}\frac{a-x}{\sqrt{a^2-x^2}}\,dx$$

$$= \int_{-a}^{a}\frac{a}{\sqrt{a^2-x^2}}\,dx - \int_{-a}^{a}\frac{x}{\sqrt{a^2-x^2}}\,dx$$

$$= a\int_{-a}^{a}\frac{1}{\sqrt{a^2-x^2}}\,dx - \int_{-a}^{a}\frac{x}{\sqrt{a^2-x^2}}\,dx$$ (1 mark)

We use the property,

$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$, if f is an even function.

$= 0$, if f is an odd function. (1 mark)

Here, $\frac{1}{\sqrt{a^2-x^2}}$ is an even function and $\frac{x}{\sqrt{a^2-x^2}}$ is an odd function.

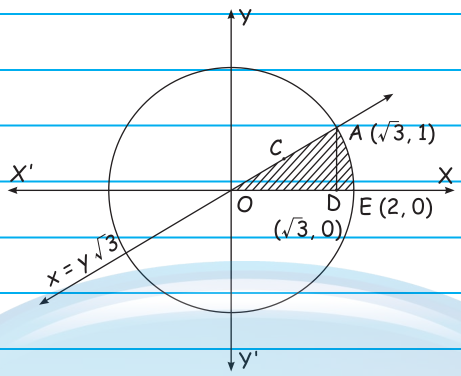
$$\therefore\ I = 2a\int_0^a \frac{1}{\sqrt{a^2 - x^2}}\,dx - 0$$

$$= 2a\left[\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$ (1 mark)

$$= 2a[\sin^{-1}1 - \sin^{-1}0]$$

$$= 2a\left(\frac{\pi}{2} - 0\right) = \pi a.$$ (1 mark)

Q. 34.



For finding the points of intersection of the circle and the line,

we solve $x^2 + y^2 = 4$ ... (1)

and $x = y\sqrt{3}$ ... (2)

From (2), $x^2 = 3y^2$

From (1), $x^2 = 4 - y^2$

$\therefore\ 3y^2 = 4 - y^2$

$\therefore\ 4y^2 = 4 \qquad \therefore\ y^2 = 1$

$\therefore\ y = 1$ in the first quadrant.

When $y = 1$, $x = 1 \times \sqrt{3} = \sqrt{3}$

$\therefore$ the circle and the line intersect at $A(\sqrt{3}, 1)$ in the first quadrant

Required area = area of the region OCAEDO

= area of the region OCADO + area of the region DAED (1 mark)

Now, area of the region OCADO

= area under the line $x = y\sqrt{3}$, i.e. $y = \frac{x}{\sqrt{3}}$ between $x = 0$ and $x = \sqrt{3}$

$$= \int_0^{\sqrt{3}} \frac{x}{\sqrt{3}}\,dx = \left[\frac{x^2}{2\sqrt{3}}\right]_0^{\sqrt{3}} = \frac{3}{2\sqrt{3}} - 0 = \frac{\sqrt{3}}{2}$$ (1 mark)

Area of the region DAED

$= $ area under the circle $x^2 + y^2 = 4$ i.e. $y = +\sqrt{4 - x^2}$ (in the first quadrant) between $x = \sqrt{3}$ and $x = 2$

$$= \int_{\sqrt{3}}^{2} \sqrt{4 - x^2}\, dx$$

$$= \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{3}}^{2}$$ (1 mark)

$$= \left[\frac{2}{2}\sqrt{4-4} + 2\sin^{-1}(1)\right] - \left[\frac{\sqrt{3}}{2}\sqrt{4-3} + 2\sin^{-1}\frac{\sqrt{3}}{2}\right]$$

$$= 0 + 2\left(\frac{\pi}{2}\right) - \frac{\sqrt{3}}{2} - 2\left(\frac{\pi}{3}\right)$$

$$= \pi - \frac{\sqrt{3}}{2} - \frac{2\pi}{3} = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$$

$\therefore$ required area $= \frac{\sqrt{3}}{2} + \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ sq units. (1 mark)